On the relationship between MOND and DM

Jörn Dunkel
Institut für Physik, Humboldt-Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany
dunkel@physik.hu-berlin.de

ABSTRACT

Numerous astrophysical observations have shown that classical Newtonian dynamics fails on galactic scales and beyond, if only visible matter is taken into account. The two most popular theoretical concepts dealing with this problem are Dark Matter (DM) and Modified Newtonian Dynamics (MOND). In the first part of this paper it is demonstrated that a generalized MOND equation can be derived in the framework of Newtonian Dark Matter theory. For systems satisfying a fixed relationship between the gravitational fields caused by DM and visible matter, this generalized MOND equation reduces to the traditional MOND law, first postulated by Milgrom. Therefore, we come to the conclusion that traditional MOND can also be interpreted as special limit case of DM theory. In the second part, a formal derivation of the Tully-Fisher relation is discussed.

Subject headings: dark matter — galaxies: kinematics and dynamics

1. Introduction

Seventy years ago, Zwicky (1933, 1937) was the first to notice that the speed of galaxies in large clusters is much too great to keep them gravitationally bound together, unless they are much heavier than one would estimate on the basis of visible matter. Since those days numerous further astrophysical observations, e.g., Doppler measurements of rotation velocities in disk galaxies, have confirmed the failure of the classical Newtonian theory, if only visible matter is taken into account (Combes et al. 1995; Bertin and Lin 1996; Field 1999; Sanders and McGaugh 2002). Historically, theoretical concepts addressing this problem can be subdivided in two categories. The first category comprises the Dark Matter (DM) theories (Binney and Tremaine 1994; Sadoulet 1999; van den Bergh 2001; Ostriker and Steinhardt 2003), whereas the second group assumes that Newton's gravitational law requires modification (Milgrom 1983a,b,c).

DM theories are based on the hypothesis that there exist significant amounts of invisible (non-baryonic) matter in the universe, interacting with ordinary visible matter only via gravity. Since empirically very successful, DM has become a widely accepted cornerstone of the contemporary cosmological standard model (Sadoulet 1999; van den Bergh 2001; Ostriker and Steinhardt 2003). Nevertheless, it must also be emphasized that until now DM has been detected only indirectly by means of its gravitational effects on the visible matter or the light.

Aiming to avoid the introduction of invisible matter, an alternative phenomenological concept was proposed by Milgrom (1983a,b,c). Instead of adapting the mass distribution, his approach requires a modified Newtonian dynamics (MOND) in the limit of small accelerations. As extensively reviewed by Sanders and McGaugh (2002), this theory can explain galaxy data, such as the flat rotation curves, in a very compelling way. On the other hand, there also have been some indications in the past that MOND might be an effective or approximate theory, applicable to a limited range of astrophysical problems only (Aguirre 2003). This hypothesis is supported by fundamental difficulties associated with relativistic generalizations of Milgrom's theory (Sanders and McGaugh 2002; Soussa and Woodard 2003; Aguirre

2003). Also, according to Aguirre et al. (2001), MOND seems to become less effective on larger scales; e.g., it cannot account for cluster densities and temperature profiles in detail.

The fact that, to some extend, both DM and MOND can successfully explain galactic dynamics favors the possibility that there exists a deeper connection between these two theories [for a general comparison, see (Aguirre 2003)]. Among others, this idea was formulated by McGaugh and de Blok (1998), and later pursued by Kaplinghat and Turner (2002). Using arguments based on galaxy formation processes in the early universe, the latter authors claim that MOND follows from cold DM theory. In his response, Milgrom (2002) questions these results. Among others, he argues that the predictions made by Kaplinghat and Turner (2002) would not only conflict with astronomical observations of pairs of galaxies (McGaugh and de Blok 1998), but also with numerical results obtained for DM models (Navarro et al. 1997). Thus, unclarity still seems to exist about whether or not MOND can in fact be understood in the framework of DM (Aguirre 2003).

It is therefore the main purpose of the present paper to explicitely demonstrate that the MOND equations (if considered as modified Newtonian gravity) can be derived from classical Newtonian dynamics, provided one also takes into account the gravitational influence of a DM component. In particular, it will be shown that the characteristic threshold acceleration, $a_0 \approx 1.2 \cdot 10^{-10}$ m/s^2, below which MOND effects begin to dominate, can also be interpreted as the asymptotic value of a more general acceleration field, characterizing the difference between the gravitational forces caused by visible matter and dark matter, respectively.

2. MOND from Newtonian dynamics with DM

As starting point, consider the Newtonian EOM of a point-like test particle

$$m\ddot{\boldsymbol{x}} = -m\nabla\left[\Phi_v(\boldsymbol{x}) + \Phi_d(\boldsymbol{x})\right], \tag{1}$$

where $\Phi_v(\boldsymbol{x})$ and $\Phi_d(\boldsymbol{x})$ denote the gravitational potentials due to visible and dark matter, respectively. Both potentials are solutions of Poisson equations,

$$\nabla^2\Phi_{v/d} = 4\pi G\ \rho_{v/d}, \tag{2}$$

where $\rho_{v/d}(\boldsymbol{x})$ is the corresponding mass density and G denotes the gravitational constant. For convenience, we define the accelerations

$$\boldsymbol{g}_{v/d}(\boldsymbol{x}) := -\nabla\Phi_{v/d}(\boldsymbol{x}). \tag{3}$$

Thus, Eq. (1) simplifies to

$$\ddot{\boldsymbol{x}} = \boldsymbol{g}_v + \boldsymbol{g}_d =: \boldsymbol{g}. \tag{4}$$

Now let us additionally assume that the acceleration vectors $\boldsymbol{g}_v$ and $\boldsymbol{g}_v$ point in the same direction, denoted by

$$\boldsymbol{g}_v \upuparrows \boldsymbol{g}_d. \tag{5}$$

Note that in this case also $\boldsymbol{g}_{v/d} \upuparrows \boldsymbol{g}$. Roughly speaking, the assumptions (5) means that the visible mass distribution ρ_v and the DM distribution ρ_d behave very similar. Next, we rewrite Eq. (4) as

$$\ddot{\boldsymbol{x}} = \left(1 + \frac{g_d}{g_v}\right)\boldsymbol{g}_v, \tag{6}$$

where $g_{v/d} := |\boldsymbol{g}_{v/d}|$ with

$$g_v = g - g_d \geq 0, \tag{7}$$

if condition (5) holds. Inserting this into (6) yields

$$\ddot{\boldsymbol{x}} \quad = \quad \left(1 + \frac{1}{g/g_d - 1}\right)\boldsymbol{g}_v. \tag{8}$$

Thus, by virtue of (4), we find that

$$\boldsymbol{g}_v = \left(\frac{\epsilon}{\epsilon + 1}\right)\boldsymbol{g} =: \tilde{\mu}(\epsilon)\boldsymbol{g}, \tag{9}$$

where we have introduced

$$\epsilon(\boldsymbol{x}) := \frac{g(\boldsymbol{x})}{g_d(\boldsymbol{x})} - 1 \geq 0. \tag{10}$$

Equation (9) can be compared to the fundamental MOND formula (Milgrom 1983a,b,c; Sanders and McGaugh 2002)

$$\boldsymbol{g}_v = \mu\left(\frac{g}{a_0}\right)\boldsymbol{g}, \tag{11}$$

where, due to empirical reasons, the function $\mu(\xi)$ is *postulated* to have the asymptotic behavior

$$\mu(\xi) = \begin{cases} 1, & \xi \gg 1; \\ \xi, & \xi \ll 1. \end{cases} \tag{12}$$

One readily observes, that this is exactly the *natural* asymptotic behavior of $\tilde{\mu}(\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$, respectively. Hence, if we identify μ with $\tilde{\mu}$ and introduce an acceleration field $a(\boldsymbol{x})$ by

$$\frac{g(\boldsymbol{x})}{a(\boldsymbol{x})} = \epsilon(\boldsymbol{x}) = \frac{g(\boldsymbol{x})}{g_d(\boldsymbol{x})} - 1, \tag{13}$$

then it becomes obvious that (9) is the natural generalization of the MOND postulate (11). The only difference is that we have a local acceleration field $a(\boldsymbol{x})$ in (9), whereas $a_0 = const$ was postulated in the MOND formula (11). Note, that Eq. (13) can also be written in the equivalent form

$$\begin{aligned} \frac{1}{a(\boldsymbol{x})} &= \frac{1}{g_d(\boldsymbol{x})} - \frac{1}{g(\boldsymbol{x})} \\ &= \frac{1}{g_d(\boldsymbol{x})} - \frac{1}{g_v(\boldsymbol{x}) + g_d(\boldsymbol{x})}. \end{aligned} \tag{14}$$

Thus, the special MOND case

$$a(\boldsymbol{x}) \equiv a_0 \tag{15}$$

implies a fixed relation between the acceleration fields due to visible and dark matter. In particular, since the characteristic MOND acceleration a_0 is relatively small, one can further infer from (14) that galaxies satisfying the MOND limit are DM dominated.

3. Axisymmetric disk galaxies and Tully-Fisher law

In the following, let us concentrate on the quasi-two-dimensional problem of axisymmetric disk galaxies. It is an experimental observation that for many such systems the Tully-Fisher relation holds (Sanders and McGaugh 2002; McGaugh and de Blok 1998)

$$v_\infty^4 := \lim_{r\to\infty} v^4(r) \propto L \propto M, \tag{16}$$

where L denotes the luminosity and M is the *visible* (baryonic) mass of the galaxy. The quantity $v(r)$ is the absolute velocity of stars or gaseous components, rotating in the disk plane around the galactic center (r is the distance from the galactic center, defining the origin of the coordinate system). Equating centripetal acceleration v^2/r and $g(r)$, we find

$$v_\infty^2 = \lim_{r\to\infty} r g(r) = \lim_{r\to\infty} r\sqrt{a(r)\, g_v(r)}. \tag{17}$$

Note that the second equality holds, only if one additionally assumes that $\epsilon(r) \ll 1$ for $r \to \infty$. The reason is that, according to (9), only in this very case the approximation $g^2 \approx a g_v$ is valid. Physically, the condition $\epsilon(r) \ll 1$ reflects a dominating DM influence, as implied by (10) and (13), respectively.

The Tully-Fisher law (16) follows directly from the rhs. of (17). Assuming that $a(r) \to a_\infty$ for $r \to \infty$ and, in agreement with the standard procedure, a Keplerian behavior $g_v(r) \simeq GM/r^2$ for $r \to \infty$, we find the desired result

$$v_\infty^4 = a_\infty GM. \tag{18}$$

For the special case $a_\infty = a_0$, this is the well-known MOND formula. Note that according to our approach Eq. (18) represents, at least formally, a derived result, whereas it plays the role of a postulate in the original MOND papers (Milgrom 1983a; Sanders and McGaugh 2002). It might be worthwhile to emphasize here once again the crucial aspect, which is that the function $\tilde{\mu}$ from (9) naturally satisfies the MOND postulates (12).

Nevertheless, one must be aware of the fact that the above derivation of Eq. (18) was essentially guided by the knowledge of the empirical Tully-Fisher law (16). More precisely, the DM paradigm in its current form does *not* provide any explanation for the fact that in many disk galaxies visible and dark matter have arranged in such a way that $a(r)$ rapidly converges to a constant non-vanishing value.

Since g_v and g_d reflect the distributions of visible and dark matter, and because of

$$\frac{1}{a_\infty} = \lim_{r\to\infty} \left\{ \frac{1}{g_d(r)} - \frac{1}{g_v(r) + g_d(r)} \right\}, \tag{19}$$

the quantity a_∞ gives us information about the asymptotic mass distributions. According to (Milgrom 1983a,b,c; Sanders and McGaugh 2002), for several disk galaxies the experimental value is given by the MOND value, $a_\infty = a_0$. From the point of view adopted in this paper, this indicates that the composition of these galaxies is generally similar.

In contrast, at least for some clusters of galaxies the actual value of $a(\boldsymbol{x})$ seems to essentially deviate from the MOND value a_0. As mentioned

earlier, Aguirre et al. (2001) have shown that the experimentally observed, radial temperature profiles of Coma, Abell 2199 and Virgo can *not* be fitted if one assumes a globally constant value $a(\boldsymbol{x}) \equiv a_0$. Furthermore, these authors report satisfactory agreement when they apply standard DM models instead. With regard to our above considerations, the latter procedure simply corresponds to using a locally varying field $a(\boldsymbol{x}) \not\equiv a_0$. On the one hand, this supports the hypothesis that MOND should be viewed as a special limit case of DM theory; on the other hand, one is led to ask, why $a(\boldsymbol{x})$ is approximately constant in disk galaxies, but seems to vary in clusters. According to the author's opinion, the answer to this question can only be given by an improved DM theory, yet to be developed. In particular, such a theory must predict the dynamics of dark and visible mass components in detail.

Finally, we still note that if $g_v(\boldsymbol{x}) \ll g_d(\boldsymbol{x})$ holds, then one can expand (14) yielding

$$a(\boldsymbol{x}) \approx \frac{g_d(\boldsymbol{x})^2}{g_v(\boldsymbol{x})}. \tag{20}$$

For spherical matter distributions this means that

$$a(r) \approx \frac{[GM_d(r)/r^2]^2}{GM_v(r)/r^2}, \tag{21}$$

where $M_{v/d}(r)$ denotes the visible/dark mass contained within radius r. For the special case $a(r) \approx a_0$, this is equivalent to

$$\frac{1}{M_v(r)}\left[\frac{M_d(r)}{r}\right]^2 \approx \frac{a_0}{G} \approx 2\,\frac{\mathrm{kg}}{\mathrm{m}^2} \approx 10^3\,\frac{\mathrm{M}_\odot}{\mathrm{pc}^2}, \tag{22}$$

which implies a strong correlation between the distributions of visible and dark matter in the MOND limit. It should be mentioned here that the possibility of such a connection was already suggested by McGaugh and de Blok (1998) and, later, also more extensively discussed by McGaugh (2000).

4. Summary and conclusions

It was shown that the generalized MOND equation (9) can be derived from Newtonian dynamics, if one adds a DM contribution Φ_d to the (baryonic) Newtonian potential Φ_v, such that $\Phi_{v/d}$ lead to equally directed accelerations $\boldsymbol{g}_{v/d} = -\nabla\Phi_{v/d}$. Compared to the traditional MOND law (11), the only formal difference consists in the fact that the constant threshold value a_0 is replaced by the more general acceleration field $a(\boldsymbol{x})$ from (14). In the DM picture, $a(\boldsymbol{x})$ reflects the local difference between the gravitational forces caused by dark and visible matter, respectively. In order to exactly regain the traditional MOND law (11), one additionally has to demand that $a(\boldsymbol{x}) \equiv a_0$. Thus, MOND can in principle also be interpreted as a DM theory, satisfying the two additional conditions (5) and (15).

Therefore, it seems reasonable to assume that the traditional MOND theory represents a special limit case of Newtonian DM theory. Adopting this point of view, one can further conclude that MOND successfully explains the rotation curves of disk galaxies because for such objects the above conditions (5) and (15) are fulfilled. If this is true, then, as also discussed above, the MOND constant a_0 can be interpreted as the asymptotic value of the field $a(r)$ as $r \to \infty$.

More generally speaking, whenever there is a fixed relationship between g_d and g_v (or ρ_d and ρ_v, respectively) such that $a(\boldsymbol{x}) \approx a_0$, then the traditional MOND theory should continue to work successfully. In turn, if a disk galaxy is in the MOND regime, then Eq. (14) can be used to estimate the DM distribution ρ_d, provided the visible matter distribution ρ_v is known from observations. Furthermore, it was shown that $\mu(\xi) = \xi/(\xi+1)$ is the natural candidate for the MOND function. Another result of this paper was the formal derivation of the Tully-Fisher law (18) in Sec. 3. This relation should hold whenever the two conditions $g_v \ll g_d$ and $a_\infty > 0$ are satisfied, where $a_\infty := \lim_{r\to\infty} a(r)$. In this context it must be stressed that the current DM model cannot explain, in which situations these two conditions are fulfilled, and, if so, why this is the case. Therefore, modifications of the conventional DM theory seem inevitably necessary.

We conclude this short paper with a more general remark. In principle, there seems to be an agreement that Newton's theory applied to visible matter does *not* give a generally correct description of the dynamics of galaxies and, therefore, has to be modified. A first way to do this is to simply consider an additional potential Φ_d and, following the standard strategy, to attach a "generating object" called DM to this potential. As shown

above, Milgrom's concept (if considered as modification of gravity) is in fact very similar, even though it seems quite different at first glance. In particular, the MOND equations can also be transformed into a modification of the former potential type, by starting with $a(\boldsymbol{x}) \equiv a_0$ and reversing the above manipulations. The "generating object" of the related potential can then be named DM as well.

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REFERENCES

Aguirre, A. 2003, to appear in ASP Conference Series: Dark Matter in Galaxies, edited by Ryder, R., Pisano, D. J., Walker, M., and Freeman, K., arXiv:astro-ph/0310572.

Aguirre, A., Schaye, J., and Quataert, E. 2001, ApJ, 561, 550

Bertin, G., and Lin, C. C. 1996, Spiral Structure in Galaxies - A Density Wave Theory, The MIT Press

Binney, J., and Tremaine, S. 1994, Galactic Dynamics, Princeton University Press

Combes, F., Boissé, P., Mazure, A., and Blanchard, A. 1995, Galaxies and Cosmology, Springer

Field, G. 1999, Rev. Mod. Phys., 71, S33

Kaplinghat, M., and Turner, M. 2002, ApJ, 569, L19

McGaugh, S. S., and de Blok, W. J. G. 1998, ApJ, 499, 41

McGaugh, S. S., and de Blok, W. J. G. 1998, ApJ, 499, 66

McGaugh, S. S. 2000, in Galaxy Dynamics: From the Early Universe to the Present, edited by Combes, F., Mamon, G. A., and Charmandaris, V., ASP 197, 153, arXiv:astro-ph/9909452

Milgrom, M. 1983, ApJ, 270, 365

Milgrom, M. 1983, ApJ, 270, 371

Milgrom, M. 1983, ApJ, 270, 384

Milgrom, M. 2002, ApJ, 571, L81

Navarro, J. F., Frenk, C. S., and White, S. D. M. 1997, ApJ, 490, 493

Ostriker, J. P., and Steinhardt, P. 2003, Science, 300, 1909

Sadoulet, B. 1999, Rev. Mod. Phys., 71, S197

Sanders, R. H., and McGaugh, S. S. 2002, ARA&A, 40, 263, arXiv:astro-ph/0204521

Soussa, M. E., and Woodard, R. P. 2001, arXiv:astro-ph/030/0307358 v1

van den Bergh, S. 2001, ASP Conference Series: Historical Development of Modern Cosmology, edited by Martinez, V. J., Trimble, V., and Pons-Borderia, M. J., 252, 75

Zwicky, F. 1933, Helv. Phys. Acta, 6, 110

Zwicky, F. 1937, ApJ, 86, 217

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